Multi-class oscillating systems of interacting neurons

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The main motivation is to find a good microscopic model to describe oscillating biological systems, mostly neurons.

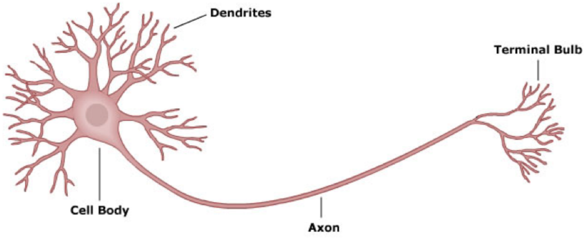
We will consider large systems of interacting point processes presenting intrinsic oscillations in large scale, although single neuron's dynamics do not encode any oscillatory behavior.

In other words : we try to answer to the following question : How does periodic behavior emerge at a macroscopic level when the single units do not have any tendency to behave periodically ?

Outline

1. Introduction of the model : Point process models for large systems of interacting neurons given by Hawkes processes.
2. Propagation of chaos for a particular multi-class system.
3. Erlang kernels allow to develop the memory. Associated Piecewise Deterministic Markov Process (PDMP).
4. Study of the oscillatory behavior of the limit system.
5. And of the finite size system $\Longrightarrow$ Large deviations.

Neurons



- Neurons : generate and propagate action potentials the long of their axons.

- They communicate by transmitting spikes : this is a fast transmembrane current of K^+/Na^+-ions, stimulated by ion pumps.

Closer look to spikes

The shape and the time duration of spikes is almost deterministic - and always "the same" (for a fixed neuron, under the same experimental conditions)

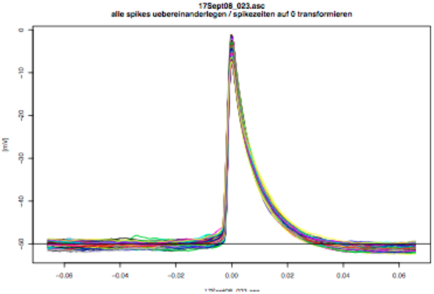
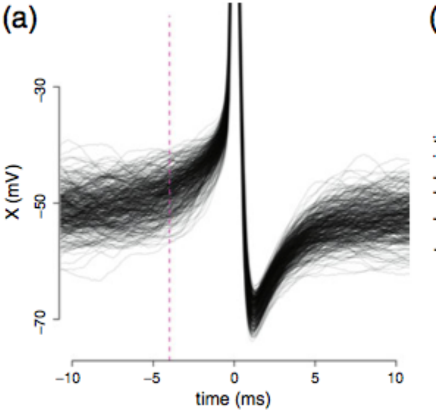


FIGURE: Picture by R. Höpfner, Mainz

The next picture is by Jahn, Berg, Hounsgaard, Ditlevsen, 2011. It also shows that spikes **do not appear** when the membrane potential hits a fixed threshold...



• The duration of each spike is very short (about 1 ms) - followed by a refractory period during which the neuron can not spike again (about 1 ms).

• Since shape of spike almost deterministic $\rightarrow$ report if at a given time there is **presence or absence of a spike** $\rightarrow$ **spike trains**.

• We do this in continuous time.

Point processes

- Point process model : for each neuron, we model the random times of appearance of a spike.
- N neurons ($=$ point processes) which interact.
- Counting process associated to neuron i, $1 \le i \le N$:

$$Z_i(t) = \text{ number of spikes of neuron } i \text{ during } [0, t].$$

- Spike counting process associated to neuron i : $Z_i(t)$ has intensity process $\lambda_i(t)$ defined by

$$P(Z_i \text{ has a jump during }]t, t+dt] | \mathcal{F}_t) = \lambda_i(t) dt.$$

- We will use Hawkes intensities : intensity $\lambda_i(t)$ incorporates the **interactions between the neurons.**
- It also represents the way the spiking behavior of a neuron depends on its history :
  It is commonly admitted that spike trains should be processes having infinite or variable memory.
- Hence $\lambda_i(t)$ is a **stochastic process, depending on the whole history before time t.**

Interacting Hawkes processes

- Intensity of $i-$th neuron given by

$$\lambda_i(t) = f_i\left(\sum_{j=1}^{N}\int_{]0,t[} h_{ij}(t-s)dZ_j(s)\right)$$

$\uparrow$ rate fct $\uparrow$ loss fct $\uparrow$ past event

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- If h_{ij} is not of compact support, then : **truly infinite memory process.**

Example

$$\lambda_i(t) = f_i\left(\sum_j \int_{]0,t[} W_{ij} e^{-\alpha_i(t-s)} dZ_j(s)\right)$$

– W_{ij} = *synaptic weight of neuron j on neuron i. If $W_{ji} > 0$, then the synapse is* **excitatory**, *if $W_{ji} < 0$, then it is* **inhibitory.**
– $e^{-\alpha_i(t-s)}$: *past events are forgotten at exponential speed.*
– *Neurons which have a direct influence on the spiking activity of i are those belonging to*

$$\mathcal{V}_i := \{j : W_{ij} \neq 0\} \quad \Rightarrow \textbf{Interaction graph}.$$

Hawkes intensity

• *Hawkes processes are very popular nowadays and widely used :*
– in neuroscience : Hansen, Reynaud-Bouret and Rivoirard (2015), Julien Chevallier (2016), ...
– in genomics : Reynaud-Bouret and Schbath (2006), ...
– in financial econometrics : Jaisson and Rosenbaum (2014), ...

• *have been introduced in 1971 by Hawkes to model earthquakes and the appearance of their aftershocks.*

• *Main idea : Self exciting (influencing) point processes : past events trigger future events.*

• *For linear Hawkes processes, there is a representation via an equivalent* **branching process.**

Discussion of the model

- $$X_i(t) := \sum_j \int_{]0,t[} h_{ij}(t-s)dZ_j(s) :$$

can be interpreted as membrane potential of neuron i at time t.

- Integrate-and-fire model : the membrane potential of neuron i collects all the past spike events of its *presynaptic neurons*. The neuron fires depending on the height of its actual membrane potential, at rate $f_i(X_i(t))$. *(Warning : in the literature, the name "Integrate-and-fire"-model is often reserved to diffusion models.)*

Multiclass systems of interacting neurons

Multi-class framework :

• Our system is made of n **populations** or clusters of neurons $k = 1, 2, \ldots, n$. This number n is **fixed** throughout the talk.

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• Each population k consists of N_k **neurons** described by their counting processes

$$Z_{k,i}(t), 1 \leq i \leq N_k.$$

• Within a population, all neurons behave in the same way. This is a mean-field assumption.

- Intensity of any neuron belonging to population k :

$$\lambda_k(t) = f_k\left(\frac{1}{N_{k+1}}\sum_{1\leq j\leq N_{k+1}}\int_{]0,t[} h_k(t-s)dZ_{k+1,j}(s)\right).$$

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- We are in a **mean field frame :** population $k+1$ influences population k only through its empirical measure. And we are in a **cyclic feedback frame** ....

Mean field limit

- What happens in the large system size limit ?

- I.e. $N = N_1 + \ldots + N_n$ total number of neurons $\to \infty$ such that for each population

$$\lim_{N\to\infty} \frac{N_k}{N} > 0.$$

- Remember the intensity of population k

$$\lambda_k(t) = f_k\left(\int_{]0,t[} h_k(t-s)\left[\frac{1}{N_{k+1}}\sum_{1\le j\le N_{k+1}} dZ_{k+1,j}(s)\right]\right)$$

$$\uparrow \text{ LLN} \to d\mathbb{E}(\bar{Z}_{k+1}(s)),$$

where $\bar{Z}_{k+1}$ is the counting process of a typical neuron belonging to population $k+1$ in the $N \to \infty-$limit.

Limit system

• Limit system : family of counting processes $\bar{Z}_k(t), k = 1, \ldots, n$, solution of an inhomogeneous equation

$$\bar{Z}_k(t) = \int_0^t \int_{\mathbb{R}_+} 1_{\{z \leq f_k(\int_0^s h_k(s-u) d\mathbb{E}(\bar{Z}_{k+1}(u))\}} N^k(ds, dz),$$

where $N^k, k = 1, \ldots, n$ are independent PRM on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dsdz$.

• Existence of a pathwise unique solution of the limit system standard under Lipschitz assumption on the f_k ; follows ideas of Delattre, Fournier and Hoffmann (2016) on high-dimensional Hawkes processes in the one-population case.

Convergence to limit system

- Convergence of the finite size system (of the collection of empirical measures of each population) to the limit : standard as well : We take empirical measures within each population and obtain

Theorem (Propagation of chaos, Ditlevsen and L. 2017)

$$\Big(\frac{1}{N_1}\sum_{1\le i\le N_1}\delta_{(Z^N_{1,i}(t))_{t\ge 0}},\ldots,\frac{1}{N_n}\sum_{1\le i\le N_n}\delta_{(Z^N_{n,i}(t))_{t\ge 0}}\Big) \to \mathcal{L}((\bar Z_1(t),\ldots,\bar Z_n(t))_{t\ge 0})$$

in probability, as $N\to\infty$. ($\mathcal{P}(D(\mathbb{R}_+,\mathbb{R}_+))$ is endowed with the weak convergence topology ass. with the Skorokhod top. on $D(\mathbb{R}_+,\mathbb{R}_+)$.)

• Multi-population frame : reminiscent of Graham (2008), see also Graham and Robert (2009) : coined the notion of "multi-chaoticity".

• Note that in the limit the different populations are independent. Interactions of classes do only survive in law.

Study of intensities of the limit system

- Taking expectations yields : $m_t^k = \mathbb{E}(\bar{Z}_k(t)), k = 1, \ldots, n$, solves

$$\frac{dm_t^k}{dt} = f_k \left(\int_0^t h_k(t-u) dm_u^{k+1} \right).$$

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- Equations depending on the whole history.

- Hawkes processes are truly infinite memory processes - the intensity depends on the whole history.

- We will present situations, in which these limit intensities $\frac{dm_t^k}{dt}$ **OSCILLATE !** We do this in the case where the system can be **completed to a system of ODE's.**

Developing the memory

- Consider Erlang memory kernels :

$$h_k(t) = c_k \frac{t^{\eta_k}}{(\eta_k)!} e^{-\nu_k t}, \nu_k > 0, \eta_k \in \mathbb{N}_0, c_k \in \mathbb{R}.$$

- The delay of influence of the past is distributed. It takes its maximum at about η_k/ν_k time units back in the past.
- The higher the order of the delay η_k, the more the delay is concentrated around its mean value $(\eta_k + 1)/\nu_k$.
- If $c_k > 0$, then the influence of pop $k+1$ on pop k is excitatory, else : inhibitory.

Recall : Limit integrated intensities given by

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CLAIM : In case of Erlang memory kernels h_k, it is possible to **complete** $(x^1, \dots, x^n)$ **to a higher dimensional system of ODE's ! ! ! !** This is a standard trick in delay equations that I am going to explain now.

Developing the memory - continued

- Suppose e.g. $h_k(t) = h(t) = c_k t e^{-\nu_k t}$ (short memory of length 1).

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- In terms of the intensity process : Introduce for $1 \le k \le n$,

$$x_t^k = \int_0^t h_k(t-s) dm_s^{k+1}, \quad y_t^k = \int_0^t h_1(t-s) dm_s^{k+1}(s).$$

$\Rightarrow$ two dimensional system of ODE's

$$\begin{aligned} \dot{x}_t^k &= -\nu_k x_t^k + y_t^k, \\ \dot{y}_t^k &= -\nu_k y_t^k + c_k \frac{dm_t^{k+1}}{dt} \end{aligned}$$

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where the last equation is linked to the next population.

Summary

- memory kernels of type $h_k(t) = c_k t e^{-\nu_k t}$ give rise to a $2n-$ dimensional system of coupled ODE's which are of type

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for $1 \leq k \leq n$.

• Increasing the delay of the memory kernel will increase the dimension of this system of coupled ODE's.

• This can be restated in terms of the original finite size jump process ....

Associated system of PDMP's

Let

$$X^{k,1}(t) = \frac{1}{N_{k+1}} \sum_{j=1}^{N_{k+1}} \int_{]0,t[} h_k(t-s) dZ_{k+1,j}(s), 1 \le k \le n,$$

and complete to system $X^{k,i}, 1 \le k \le n, 1 \le i \le \eta_k + 1$: PDMP with generator

$$A\varphi(x) = \sum_{k=1}^{n} \left[\sum_{i=1}^{\eta_k} \{-\nu_k x^{k,i} + x^{k,i+1}\} \frac{\partial \varphi}{\partial x^{k,i}} - \nu_k x^{k,\eta_k+1} \frac{\partial \varphi}{\partial x^{k,\eta_k+1}} \right] + \sum_{k=1}^{n} N_{k+1} f_{k+1}(x_{k+1,1}) \left[\varphi(x + \frac{c_k}{N_{k+1}} e_{k,\eta_k+1}) - \varphi(x) \right].$$

Some simulations in the case of a single neuron

A single neuron's spike train represented by a Hawkes process with an Erlang memory kernel, of memory order 3 :

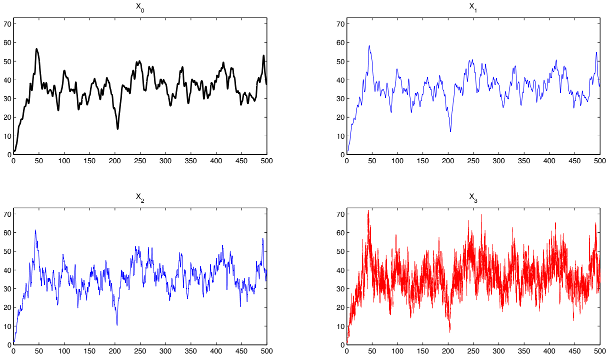

FIGURE: Picture by Aline Duarte, Sao Paulo

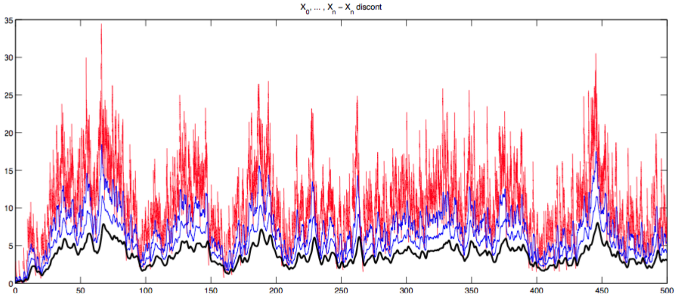

FIGURE: Picture by Aline Duarte

Monotone cyclic feedback systems

- Recall we wanted to find oscillations for the limit intensities.
- Our system of coupled ODE's in case of memory of order 1 : For $1 \leq k \leq n$,

$$\dot{x}_t^k = -\nu_k x_t^k + y_t^k, \quad \dot{y}_t = -\nu_k y_t^k + c_k f_{k+1}(x_t^{k+1}).$$

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- This system is a **monotone cyclic feedback system** (Mallet-Paret and Smith 1990).

– Cyclic means : population k is only influenced by population $k+1$, for all k.
– Feedback : population n is influenced by population 1.
– Monotone : all rate functions f_k are non-decreasing.

• Put $\delta := \prod_{k=1}^n c_k$. If $\delta > 0$, the system is of positive feedback, else, it is of negative feedback.

• Put $\delta := \prod_{k=1}^n c_k$. If $\delta > 0$, the system is of positive feedback, else, it is of negative feedback. We will consider the **negative feedback case.**

Suppose that $f_k, 1 \leq k \leq n$, are bounded analytic Lipschitz functions and that the system is of negative feedback. Then :

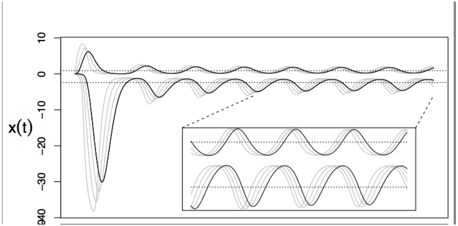
Theorem (Mallet-Paret and Smith)

1) $\exists!$ equilibrium point x^* of the above system.
2) $\exists$ easily verifiable condition implying that x^* is unstable. In this case, there exists at least one – but not more than a finite number of – non constant periodic orbits. One of them is attracting.

Remark

So here they are, the oscillations (not for the m_t^k, but for the intensities)! Because : non constant periodic orbit = oscillations

Simulation of a system with 2 populations and memory 3 for the first population and memory 4 for the second one :



The role of the order of the memory

Definition

We call order of the memory of population k the index $\eta_k \in \mathbb{N}$ such that

$$h_k(t) = c_k \frac{t^{\eta_k}}{(\eta_k)!} e^{-\nu_k t}.$$

We call "total order of memory of the system" the number $\kappa := n + \sum_{k=1}^{n} \eta_k$.

Proposition (Hopf bifurcation due to increasing memory)

Suppose that $\nu_k = 1$, for all $1 \leq k \leq n$. Then there exists κ^* such that for all $\kappa < \kappa^*$, the equilibrium point x^* is stable. For $\kappa \geq \kappa^*$, the systems presents oscillations.

So increasing the DELAYS pushes the system towards oscillations.

Central limit Theorem

We have well understood the behavior of the limit system ....

To which extent does the large time behavior of the limit system $(m_t^1, \ldots, m_t^n)$ **predict** *the large time behavior of the finite size system ? ? ?*

$\Rightarrow$ CLT where convergence of both N and t to infinity is considered. Under suitable assumptions on the way $N, t \to \infty$: depends on spectral properties of offspring matrix.

Diffusion approximation of the intensity process

Second answer to : To which extent are the oscillations of the limit system also felt by the finite size system ? :
Have a look at the "Large intensity-small jump size"-diffusion approximation (in case $n = 2$ and $\eta_1 = \eta_2 = 1$) :

Recall the generator of the associated PDMP :

$$A\varphi(x) = \sum_{k=1}^{2} \left[\{-\nu_k x^{k,1} + x^{k,2}\} \frac{\partial \varphi}{\partial x^{k,1}} - \nu_k x^{k,2} \frac{\partial \varphi}{\partial x^{k,2}} \right] + \sum_{k=1}^{2} N_{k+1} f_{k+1}(x_{k+1,1}) \left[\varphi(x + \frac{c_k}{N_{k+1}} e_{k,2}) - \varphi(x) \right].$$

Small jumps of size $\frac{c_k}{N_{k+1}}$ appearing at rate $N_{k+1} f_{k+1}$ $\Rightarrow$

$$\left\{ \begin{array}{rcl} dX_1(t) & = & -\nu_1 X_1(t)dt + Y_1(t)dt \\ dY_1(t) & = & -\nu_1 Y_1(t)dt + c_1 f_2(X_2(t))dt \\ & & + \frac{c_1}{\sqrt{N_2}} \sqrt{f_2(X_2(t))} dB_2(t) \end{array} \right\},$$

similar equations for the 2nd population $(X_2(t), Y_2(t))$.

- Can be extended to higher order delays in Erlang memory kernels $\Longrightarrow$ longer cascades of SDE's.

- We have the control on the weak error

$$\|P_t\varphi - \tilde{P}_t\varphi\|_\infty \leq Ct\frac{\|\varphi\|_{4,\infty}}{N^2}.$$

General comments

• We obtain a diffusion of high dimension driven by only 2 Brownian motions - each of them approximating the jump noise of one of the populations.

• We have to treat the memory terms as auxiliary variables. This gives rise to coordinates of the diffusion without noise $\Rightarrow$ Highly degenerate diffusion.

• **Cascade structure** of the drift : a coordinate does only depend on itself and the following coordinate.

- Due to the **cascade structure** of the drift it is easy to show that the diffusion satisfies the weak Hörmander condition.
- Hence it is strong Feller (*Ichihara and Kunita 1974*).

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• Using a convenient Lyapunov-function and the control theorem

$\Longrightarrow$ $\exists$ attainable point (which can be chosen to be the unstable equilibrium of the limit monotone cyclic feedback system).

$\Longrightarrow$ diffusion is recurrent in the sense of Harris, with unique invariant probability measure.

Theorem

Let Γ be a non constant periodic orbit of the limit system which is asymptotically orbitally stable. Then for all $\varepsilon > 0$ and for all $T > 0$, for all starting configurations x, P_x−almost surely,

the approx diffusion visits $B_\varepsilon(\Gamma)$ during a time period of length T,

infinitely often.

Hence the diffusion approximation visits the oscillatory region infinitely often.

Large deviations

• **Large deviations result :** For large N, the diffusion stays within tubes around the limit cycle during long periods, before eventually leaving such a tube after a time which is of order

$$e^{N\bar{V}},$$

$\bar{V}$: quasi-potential, related to control problem : cost of steering the process from the limit cycle to the boundary of the tube around the limit cycle.

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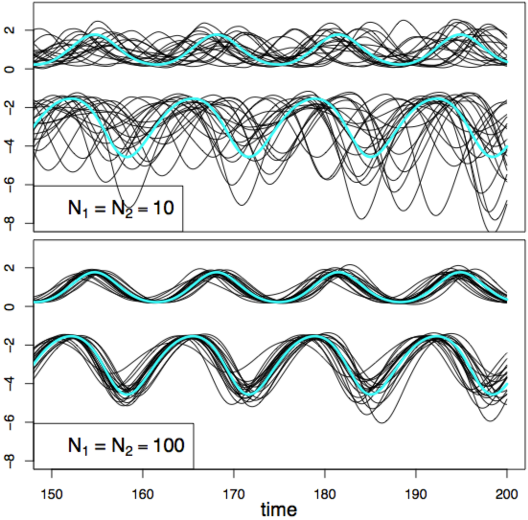
• Can be made precise in the sense of sample path large deviations for diffusions with small noise, in the sense of Freidlin-Wentzell (*although diffusion is highly degenerate*). Most important point : establish the **necessary control theory** in our framework. See Löcherbach JTP 2017 for details.

Some simulations of the approximating diffusion in the case $n = 2$



Conclusions

• Infinite memory (of Hawkes processes) and introduction of successive memory terms as auxiliary variables give rise to hypo-elliptic diffusion approximation and its specific **cascade** structure.

• This cascade structure implies :
– weak Hörmander condition
– controllability of the system

• Oscillations appear from the non-linear "McKean-Vlasov"-type structure of the limit system (system whose dynamics depends on its own law) - the dynamics of each single particle do not include any periodic behavior.

Remarks

- Specific interaction graph structure not necessary for propagation of chaos : other interaction graphs possible.
- What happens if there are periodic changes in the underlying interaction graph ?
- Example of a dynamical system where there are several coexisting stable orbits ?
- What happens when the synaptic strength (i.e. the factor c_k) changes over time ($\rightarrow$ plasticity ?)
- And if we add an external signal during some time ?

Final remarks on Hawkes processes

- Erlang kernels allow to describe certain Hawkes processes via an associated system of PDMP's
- Their stability behavior can be easily analyzed.
- Gives another approach to Simulation and Stability of non-linear Hawkes processes (work with A. Duarte and G. Ost, 2017).

Some literature

• Collet, F., Dai Pra, P., Formentin, M. Collective periodicity in mean-field models of cooperative behavior. *Nonlinear Differential Equations and Applications NoDEA*, 22(5) (2015).

• Delattre, S., Fournier, N., Hoffmann, M., High dimensional Hawkes processes. *Ann. App. Probab.* 26 (2016). 216–261

• Galves, A., L. E., Modeling networks of spiking neurons as interacting processes with memory of variable length. *Journal de la Société Française de Statistiques* 157 (2016).

• Graham, C. Chaoticity for multiclass systems and exchangeability within classes. *J. Appl. Prob.*, 45 (2008).

• HANSEN, N., REYNAUD-BOURET, P., RIVOIRARD, V. Lasso and probabilistic inequalities for multivariate point processes. *Bernoulli*, 21(1) (2015).

• MALLET-PARET, J., SMITH, H.L. The Poincaré-Bendixson Theorem for Monotone Cyclic Feedback Systems. *J. of Dynamics and Diff. Equations 2*, 4 (1990).

• SCHEUTZOW, M. Some examples of nonlinear diffusion processes having a time-periodic law. *The Annals of Probability*, 13(2) (1985).

Paper is on arXiv :

• DITLEVSEN, S. LÖCHERBACH, E. Multi-class oscillating systems of interacting neurons. SPA 2017, Available on http ://arxiv.org/abs/1512.00265, 2015.

• DUARTE, A., LÖCHERBACH, E., OST, G. Stability, convergence to equilibrium and simulation of non-linear Hawkes Processes with memory kernels given by the sum of Erlang kernels. Submitted, available on arxiv.org/abs/1610.03300

• LÖCHERBACH, E. Large deviations for oscillating systems of interacting Hawkes processes in a mean field frame. JTP 2017, arxiv.org/abs/1709.09356.

Thank you for your attention.