A CBI approach to financial Modelling

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Introduction

- Particular behaviors of some financial data:
  - Sovereign bond markets with persistency of low interest rates and significant fluctuations in the Euro zone;
  - Electricity prices exhibit high spikes and rapid mean-reversion, seasonality...
- Self-exciting features and jump clustering effect?
- How to include all the features into a unified and parsimonious framework description?
- An approach based on CBI (continuous state branching processes with immigration) processes

Figure: 10-years interest rates of Euro area countries.

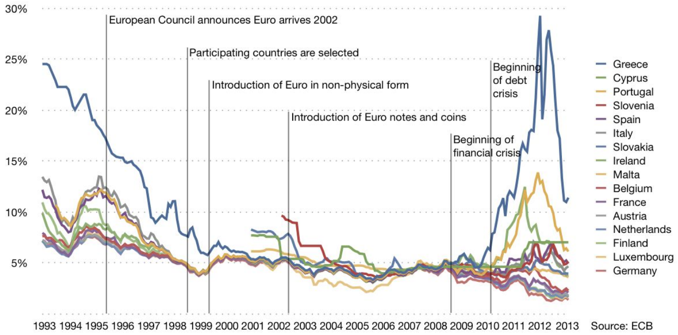
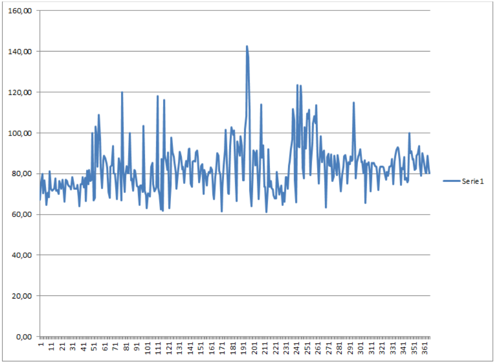


Figure: Daily electricity prices in Italy on 2012.



Modelling approaches in finance

- Hawkes process to model the "self-exciting" and the "clustering" feature: Aït-Sahalia & Jacod (2009), Errais, Giesecke & Goldberg (2010), Dassios & Zhao (2011), Rambaldi, Pennesi & Lillo (2014), and Jaisson & Rosenbaum (2015)...
- Affine models for interest rate term structure: Duffie, Pan & Singleton (2000), Filipović (2001, 2009), Duffie, Filipović & Schachermayer (2003), Keller-Ressel & Steiner (2008), ...
- Random fields description in interest rate and energy: Kennedy (1994), Albeverio, Lytvynov & Mahnig (2004), Benth, Kallsen, Meyer-Brandis (2007), Barndorff-Nielsen, Benth &Veraart (2013)

Some literature on CBI processes

Books

- Li, Z.: *Measure-Valued Branching Processes*, Springer, Berlin (2011).
- Pardoux, E.: *Probabilistic Models of Population Evolution*, Springer, Berlin (2016)

(Very partial) Papers

- Dawson, D.A. & Li, Z.: Skew convolution semigroups and affine Markov processes. *Ann. Probab.* 34, 1103-1142 (2006)
- Dawson, A. & Li, Z.: Stochastic equations, flows and measure-valued processes. *Ann. Probab.* 40, 813-857 (2012)
- Li, Z. & Ma, C.: Asymptotic properties of estimators in a stable Cox-Ingersoll-Ross model. *Stoch. Proc. Appl.* 125, 3196-3233 (2015)

Plan of the talk

- Continuous state branching processes
- α-CIR model and properties
- Applications to interest rate and power price modelling
- Concluding remarks

Model formulation

Integral representation

$$\begin{aligned} Y_t = Y_0 + \int_0^t a\,(b - Y_s)\,ds + \sigma \int_0^t \int_0^{Y_s} W(ds, du) \\ + \sigma_Z \int_0^t \int_0^{Y_{s-}} \int_{\mathbb{R}^+} \zeta \widetilde{N}(ds, du, d\zeta), \end{aligned} \tag{1}$$

- $W(ds, du)$: white noise on $\mathbb{R}_+^2$ with intensity $dsdu$,
- $\widetilde{N}(ds, du, d\zeta)$: compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $dsdu\mu(d\zeta)$,
- $\mu(d\zeta)$ is a Lévy measure satisfying $\int_0^\infty (\zeta \wedge \zeta^2)\mu(d\zeta) < \infty$. Besides, W and N are independent of each other.
- It follows from of Dawson and Li (2012) that this equation has a unique strong solution.

The self-exciting feature

- We want to illustrate how the self-exciting property arises in the present framework.
- Consider a simple Hawkes process with exponential kernel, which is defined as a point process J with intensity

$$Y_t = Y_t^* + \int_0^t e^{-a(t-s)} dJ_s \tag{2}$$

  and Y^* is deterministic, representing the background rate.
- When a jump arrives, the intensity increases, which incites the arrival of the next jump, that is the so-called self-exciting property of Hawkes processes.

Link to Hawkes process

- In order to facilitate the comparison with our integral representation, we give a different characterization of the intensity.
- Let N be a Poisson random measure on $\mathbb{R}_+^2$ with intensity $dsdu$. Consider the case where J_t is of the form $\int_0^t \int_0^{Y_{s-}} N(ds, du)$ and hence

$$Y_t = Y_t^* + \int_0^t \int_0^{Y_{s-}} e^{-a(t-s)} N(ds, du). \tag{3}$$

- In this form, the self-exciting feature can be observed as follows: the frequency of jumps grows with the process itself due to the presence of the integral with respect to the variable u. Moreover, when Y^* takes certain particular form, Y becomes a branching process.
- In this context, the self-exciting features is equivalent to the branching property and the jump intensity is proportional to the process Y itself.

Link to CIR model

- A particular case when the jump term vanishes corresponds to the well-known CIR model for short interest rates r_t.
- We illustrate the connection of the above integral representation for the CIR model with Hawkes processes.
- When $\sigma_Z = 0$, the CIR process r is given in the form:

$$r_t = r_0 + \int_0^t a\,(b - r_s)\,ds + \sigma \int_0^t \int_0^{r_s} W(ds, du), \qquad (4)$$

- The equivalent form is

$$r_t = r_t^* + \sigma \int_0^t \int_0^{r_s} e^{-a(t-s)} W(ds, du) \qquad (5)$$

where r_t^* is a deterministic function given by $r_t^* = r_0 e^{-at} + ab \int_0^t e^{-a(t-s)} ds$. This expression shows the self-exciting feature.

Link to Hawkes process (continued)

- When $\sigma = 0$ and $\mu(d\zeta) = \delta_1(dz)$, then Y is given by

$$Y_t = Y_0 + abt - \int_0^t (a + \sigma_N) Y_s ds + \sigma_N \int_0^t \int_0^{Y_{s-}} N(ds, du) \quad (6)$$

which is the intensity of Hawkes process $\int_0^t \int_0^{Y_{s-}} N(ds, du)$, N being the Poisson random measure with intensity $dsdu$.

- Consider a sequence $\left\{ Y_t^{(n)}, t \geq 0 \right\}_{n \geq 1}$ defined by (6) with parameters $(a/n, nb, \sigma_N)$. Then

$$(Y_{nt}^{(n)}/n, t \geq 0) \xrightarrow{\mathcal{L}} r \quad \text{in } D(\mathbb{R}_+),$$

where $D(\mathbb{R}_+)$ is the Skorokhod space of càdlàg processes and the process r follows a CIR model.

- Jaisson and Rosenbaum (2015): nearly unstable Hawkes process converges, after suitable scaling, to a CIR process.

The α-CIR model setup

We consider the root SDE representation of the α-CIR model

$$r_t = r_0 + \int_0^t a\left(b - r_s\right) ds + \sigma \int_0^t \sqrt{r_s} dB_s + \sigma_Z \int_0^t r_{s-}^{1/\alpha} dZ_s \qquad (7)$$

- $B = (B_t, t \geq 0)$ a Browinan motion
- $Z = (Z_t, t \geq 0)$ a spectrally positive α-stable compensate Lévy process with parameter $\alpha \in (1, 2]$ with

$$\mathbb{E}\left[e^{-qZ_t}\right] = \exp\left\{-\frac{tq^\alpha}{\cos(\pi\alpha/2)}\right\}, \quad q \geq 0.$$

- B and Z are independent

Z_t follows the α-stable distribution $S_\alpha(t^{1/\alpha}, 1, 0)$ with scale parameter $t^{1/\alpha}$, skewness parameter 1 and zero drift.

Equivalence of two representations

We choose the Lévy measure in the integral representation to be

$$\mu(d\zeta) = -\frac{1_{\{\zeta>0\}}d\zeta}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}, \quad 1 < \alpha < 2, \tag{8}$$

Then the root representation (7) and the integral representation (1) are equivalent in the following sense by Li (2011):

- The solutions of the two equations have the same probability law.
- On an extended probability space, they are equal almost surely.

A natural extension of the CIR model

- When $\sigma_Z = 0$, we recover the CIR model.
- When $\alpha = 2$, it also reduces to a CIR model but with volatility parameter $(\sigma^2 + 2\sigma_Z^2)^{1/2}$.
- The difference of Z from a Brownian motion is controlled by the tail index α:
  ⋄ $\alpha = 2$: Z is a Brownian motion scaled by $\sqrt{2}$;
  ⋄ $\alpha < 2$: Z is a pure jump process with heavy tails. More as α close to 1, more likely Z_t takes values far from median;
  ⋄ comparison with Poisson process: Z has an infinite number of (small) jumps over any time interval, allowing it to capture the extreme activity.
- Existence of the unique strong solution by Fu and Li (2010).

Simulation of processes Z and r with different α

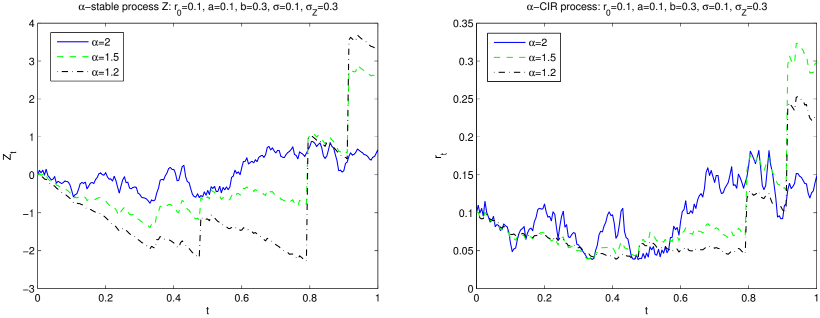


Figure: Three parameters of α: 2 (blue), 1.5 (green) and 1.2 (black)

Similar properties with CIR model

Boundary condition:

The point 0 is an inaccessible boundary if and only if $2ab \geq \sigma^2$. In particular, a pure jump α-CIR process with $ab > 0$ never reaches 0 since $\sigma = 0$.

Branching property :

r can be decomposed as $r = r^{(1)} + r^{(2)}$ where for $i = 1, 2$, $r^{(i)}$ is an α-CIR$(a, b^{(i)}, \sigma, \sigma_Z, \alpha)$ process such that $r_0 = r_0^{(1)} + r_0^{(2)}$ and $b = b^{(1)} + b^{(2)}$.

Continuous state branching process with immigration (CBI)

CBI (Kawazu & Watanabe 1971) of branching mechanism $\Psi(\cdot)$ and immigration rate $\Phi(\cdot)$: Markov process X with state space $\mathbb{R}_+$ verifying

$$\mathbb{E}_x\left[e^{-pX_t}\right] = \exp\left[-xv(t,p) - \int_0^t \Phi\big(v(s,p)\big)ds\right],$$

where $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\frac{\partial v(t,p)}{\partial t} = -\Psi(v(t,p)), \quad v(0,p) = p$$

and Ψ and Φ are functions on $\mathbb{R}_+$ given by

$$\Psi(q) = \beta q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(du),$$
$$\Phi(q) = \gamma q + \int_0^\infty (1 - e^{-qu})\nu(du),$$

with $\sigma, \gamma \geq 0$, $\beta \in \mathbb{R}$ and π, ν being two Lévy measures such that $\int_0^\infty (u \wedge u^2)\pi(du) < \infty$ and $\int_0^\infty (1 \wedge u)\nu(du) < \infty$.

Link between α-CIR and CBI processes

Let r be an α-CIR $(a, b, \sigma, \sigma_Z, \alpha)$ process. Then r is a CBI with

$$\text{branching mechanism: } \Psi(q) = aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha \quad (9)$$

$$\text{immigration rate: } \Phi(q) = abq. \quad (10)$$

Consequences:

- Let $r^{(\alpha)}$ be α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process, $\alpha \in (1, 2]$. Then $r^{(\alpha)} \xrightarrow{\mathcal{L}} r^{(2)}$ in $D(\mathbb{R}_+)$ as $\alpha \to 2$.
- Laplace transform (cf. Filipović (2001)):

$$\mathbb{E}\Big[e^{-\xi r_t - p\int_0^t r_s ds}\Big] = \exp\Big(- r_0 v(t,\xi,p) - \int_0^t \Phi\big(v(s,\xi,p)\big)ds\Big),$$

$$\text{with } \partial_t v(t,\xi,p) = -\Psi(v(t,\xi,p)) + p, \quad v(0,\xi,p) = \xi.$$

- As $t \to +\infty$, r_t has a limite distribution r_∞ given by

$$\mathbb{E}[e^{-pr_\infty}] = \exp\left\{-\int_0^p \frac{\Phi(q)}{\Psi(q)}dq\right\}, \quad p \geq 0.$$

Equivalent martingale measure for bond pricing

- Let r be an α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ processes under the initial probability $\mathbb{P}$.
- Fix $\eta \in \mathbb{R}$ and $\theta \in \mathbb{R}_+$, and define

$$U_t := \eta \int_0^t \int_0^{r_s} W(ds, du) + \int_0^t \int_0^{r_{s-}} \int_0^\infty (e^{-\theta\zeta} - 1) \widetilde{N}(ds, du, d\zeta).$$

- Change of probability: $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(U)$, with $\mathcal{E}(U)$ the Doléans-Dade exponential of U (Kallsen & Muhle-Karbe, 2010).
- r is an α-CIR$(a', b', \sigma, \sigma_Z, \alpha)$ type process under $\mathbb{Q}$ with

$$a' = a - \sigma\eta - \frac{\alpha\sigma_Z}{\cos(\pi\alpha/2)}\theta^{\alpha-1}, \quad b' = ab/a',$$

and a modified Lévy measure

$$\mu'(d\zeta) = -\frac{e^{-\theta\zeta} 1_{\{\zeta>0\}}}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}} d\zeta.$$

r remains to be a CBI process under $\mathbb{Q}$.

Application to bond pricing

For simplicity, we assume that the short rate r is given by an α-CIR$(a, b, \sigma, \sigma_Z, \mu, \alpha)$ model under $\mathbb{Q}$.

- Zero-coupon bond price:

$$B(t,T) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s ds\Big)\,\big|\,\mathcal{F}_t\Big] = \exp\Big(-r_t v(T-t) - ab\int_0^{T-t} v(s)ds\Big)$$

  where $v(\cdot)$ is given by

$$\frac{\partial v(t)}{\partial t} = 1 - \Psi(v(t)), \quad v(0) = 0,$$

  with $\Psi(q) = aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha$.

- We have

$$v(t) = f^{-1}(t) \text{ where } f(t) = \int_0^t \frac{dx}{1-\Psi(x)} \qquad (11)$$

Proposition

The function $v(\cdot)$ is increasing with respect to $\alpha \in (1, 2]$. In particular, the bond price $B(0, T)$ is decreasing with respect to α.

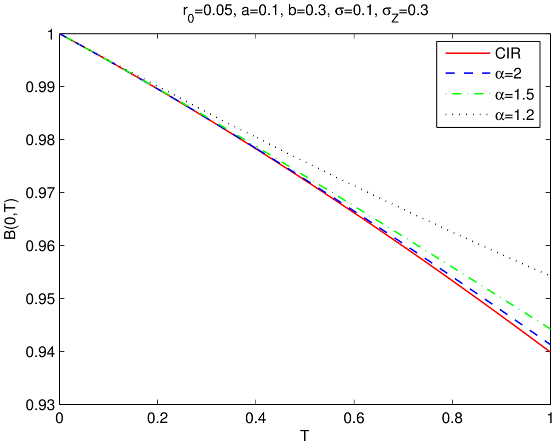


Figure: Bond price is decreasing w.r.t. α, curve CIR (in red) corresponds to $\sigma_Z = 0$

Remarks on bond prices

- Empirical studies underline that CIR model systematically overestimates short interest rates, e.g. Brown and Dybvig (1986) and Gibbons and Ramaswamy (1993)
- The above proposition shows that the α-CIR model is suitable to describe the low interest rate in the expectation sense.
- Explanation based on self-exciting property: as the interest rate becomes low, the self-exciting feature implies decreasing frequency of jumps and enforce the tendency of low interest rate.
- In other CIR+jump models e.g. Duffie and Gârleanu (2001), Keller-Ressel and Steiner (2008), LOU etc., the bond prices are in general smaller than the CIR ones (difficult to reconcile the jumps with low interest rate).

Jump behavior

- The jumps, especially the large jumps capture the significant changes in the interest rate and may imply the downgrade risk of credit quality.
- Fix $y > 0$. Consider the jumps of the process r which are larger than $\sigma_Z y$ and the associated truncated process $r^{(y)}$ as

$$r_t^{(y)} = r_0 + \int_0^t \widetilde{a}(\alpha, y)\big(\widetilde{b}(\alpha, y) - r_s\big)ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) \\ + \sigma_Z \int_0^t \int_0^{r_{s-}} \int_0^y \zeta \widetilde{N}(ds, du, d\zeta).$$

- It is also a CBI process which coincides with r up to the first large jump $\tau_y := \inf\{t > 0 : \Delta r_t > \sigma_Z y\}$ and has the branching mechanism given by

$$\Psi^{(y)} = \Psi + \sigma_Z^\alpha \int_y^\infty (1 - e^{-q\zeta})\mu(d\zeta).$$

Laplace transform of the jump counter process

Let J_t^y denote the number of jumps of r with jump size larger than $\sigma_Z y$ in $[0, t]$, i.e.

$$J_t^y := \sum_{0 \le s \le t} 1_{\{\Delta r_s > \sigma_Z y\}}.$$

Then for any $p \ge 0$ and $t \ge 0$,

$$\mathbb{E}\left[e^{-pJ_t^y}\right] = \exp\left(-l(p, y, t)r_0 - ab \int_0^t l(p, y, s)ds\right)$$

where $l(p, y, t)$ is the unique solution of the following equation

$$\frac{\partial l(p, y, t)}{\partial t} = \sigma_Z^\alpha \int_y^\infty \left(1 - e^{-p - l(p,y,t)\zeta}\right)\mu_\alpha(d\zeta) - \Psi_\alpha^{(y)}(l(p, y, t)),$$

with initial condition $l(p, y, 0) = 0$.

Probability law of the first large jump

We have

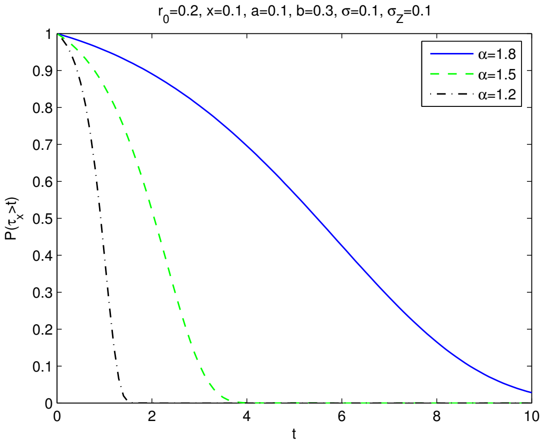
$$\mathbb{P}(\tau_y > t) = \mathbb{P}(J_t^y = 0) = \exp\Big(- I(y,t) r_0 - ab \int_0^t I(y,s) ds \Big)$$

where $I(y,t)$ is the unique solution of

$$\frac{dI}{dt}(y,t) = \sigma_Z^{\alpha} \int_y^{\infty} \mu(d\zeta) - \Psi^{(y)}(I(y,t)),$$

with initial condition $I(y,0) = 0$.

Probability function $\mathbb{P}(\tau_y > t)$ for the first big jump



Application to power price modeling

- We assume the spot price process S_t to evolve accordng to the following dynamics:

$$S_t = \alpha(t) + X_t$$

where $\alpha(t)$ is a seasonality function of deterministic type and the process X_t is a superposition of the factors Y_t^i:

$$X_t = \sum Y_t^i,$$

- The factors Y_t^i evolve according to equation (1) written before, but we neglect the Brownian contribution

$$Y_t^i = Y_0^i + \int_0^t a_i \left(b_i - Y_s^i \right) ds + \sigma_i \int_0^t \int_0^{Y_{s-}^i} \int_{\mathbb{R}^+} \zeta \widetilde{N}_i(ds, du, d\zeta)$$

- Since $\nu_i(ds, du, d\zeta) = dsdu\widetilde{\nu}_i(d\zeta)$, we can write

$$Y_t^i = Y_0^i - A_i \int_0^t (B_i - Y_s^i) ds + \sigma_i \int_0^t \int_0^{Y_{s-}^i} \int_{\mathbb{R}^+} \zeta N_i(ds, du, d\zeta)$$

  where

$$A_i = a_i - \sigma_i \int_{\mathbb{R}^+} \zeta \widetilde{\nu}_i(d\zeta), \quad B_i = \frac{a_i b_i}{A_i}$$

  with $\widetilde{N}_i(ds, du, d\zeta)$ being the compensated measure of a compound Poisson process with positive jumps.
- This kind of dynamics extends that proposed by Benth, Kallsen & Meyer-Brandis (2007), by keeping the basic features of an Ornstein-Uhlenbeck process driven by a subordinator, but it introduces the self-exciting properties in a direct and natural way.

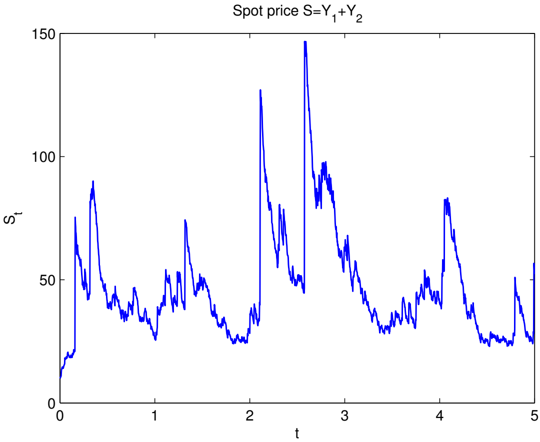


Figure: The Power Spot Price Dynamics.

Derivatives pricing

- Similar as for the interest rate modelling, we can define the equivalent probability measures $\mathbb{Q}$ and the spot process Y remains to be in the class of integral type processes.
- In the present model framework, the Forward contract $F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t]$ can be computed explicitly and so are the Flow Forwards

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1}\mathbb{E}^{\mathbb{Q}}\Big[\int_{T1}^{T2} S_u du \,|\, \mathcal{F}_t\Big]$$

- It is possible to obtain in an almost closed-form the prices of European options written on Forward contracts by using the Laplace transform of each factor.

The risk premium

- The risk premium is a relevant quantity in power markets description defined by

$$R(t,T) = \mathbb{E}^{\mathbb{Q}}\left[S_T|\mathcal{F}_t\right] - \mathbb{E}^{\mathbb{P}}\left[S_T|\mathcal{F}_t\right]$$

- We provide an explicit representation for this quantity which exhibits the sign change feature discussed in literature.

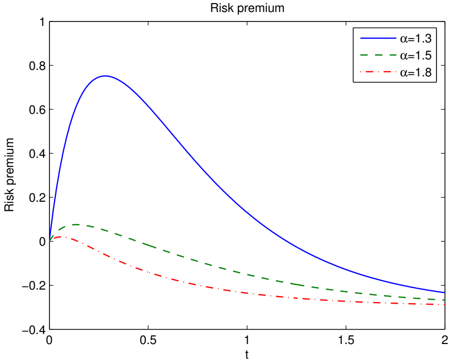


Figure: The Risk Premium Term Structure.

Concluding remarks

- The model framework just presented can include all the basic features of interest rate and power price dynamics.
  - It characterizes in a natural and parsimonious way the self-exciting property.
  - It allows to obtain in a closed form the prices of the most common derivatives
  - It exhibits some interesting features observed on the markets.
- Future perspectives include a systematic empirical investigation on real data and efficient calibration techniques.

Upcoming workshop

"Branching processes and related topics"

East China Normal University

Shanghai, China
21 – 25 May, 2018

Thanks for your attention !